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THE EXISTENCE AND CALCULATION OF SOLUTIONS OF CERTAIN INTEGRO-DIFFERENTIAL EQUATIONS IN SEVERAL DIMENSIONS

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THE EXISTENCE AND CALCULATION OF SOLUTIONS OF CERTAIN INTEGRO-DIFFERENTIAL EQUATIONS IN SEVERAL DIMENSIONS

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ABSTRACT: Boundary and initial value problems are solved for linear integro-differential equations in several dimensions of a type including linearized forms of Boltzmann's equation. The methods of calculation are based on finite differences and are thus suitable for calculation.

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The Existence and Calculation of Solutions of Certain Integro-Differential Equations in Several Dimensions

This report is concerned with boundary-, initial-value problems for certain linear integro-differential equations including linearized forms of Boltzmann's equation. A particular finite difference procedure is proved to converge; the scheme is the same as that in use at the David Taylor Model Basin.

This work was carried out under NOL Task No. FR-30.

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By direction

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THE EXISTENCE AND CALCULATION OF SOLUTIONS OF CERTAIN INTEGRO-DIFFERENTIAL EQUATIONS IN SEVERAL DIMENSIONS

INTRODUCTION

In this paper, boundary and initial value problems will be solved for linear integro-differential equations, in several dimensions, of a type including linearized forms of Boltzmann's equation. The methods of solution are based on finite differences and are thus suitable for calculation.

These methods are indirect, for our difference scheme is proved to converge, not for the original problem, but for a modification of it, whose solution approximates that of the original. A better difference scheme, or a better proof of convergence, might eliminate the need to modify the original problem. This need to modify the problem, however, may be real, desirable in numerical practice as well as in theory.

Most discussions of the integro-differential equations treated here occur in the context of neutron transport theory, of which a comprehensive account is to be had in B. Davison's book [1]. Calculation techniques in transport theory are also surveyed in R. D. Richtmyer's book [2] and in a subsequent paper [3] of E. H. Bareiss. The first proof that general multi-dimensional problems resembling ours have solutions was given by semi-group methods by K. Jörgens [4]. We do not refer here to the extensive literature on problems in one or two dimensions.

1. Statement of problem. Notation. Main Results. Let

$$\mathfrak{X}: \quad |x_r| \leq C, \qquad r = 1, \ldots, d,$$

designate a d-dimensional cube of edge 2C, the symbol $x=(x_1,x_2, \ldots,x_d)$ referring to any of its points. Let $y = (y_1, \ldots,y_d)$ and $y' = (y_1',\ldots,y_d')$ denote any d-dimensional vectors and $dy' = dy_1' \ldots dy_d'$ the d-dimensional element of volume, and let t be an independent variable on the semi-axis $t \geq 0$. The equations considered are those of the form

$$1.1 \qquad u_t + \sum_{r=1}^{d} y_r u_{x_r} + c(x,y,t)u = \int K(x,y,t,y')\, u(x,y',t)dy' + g(x,y,t)$$

with coefficient $c(x,y,t)$ and inhomogeneous part $g(x,y,t)$ defined in a (2n+1)-dimensional cylindrical drum,

$$S_T: \quad x \in \mathfrak{X}, \quad 0 \leq t \leq T, -\infty < y_s < \infty \quad (T=\text{const.} > 0), \quad s = 1,\ldots,d$$

and with kernel $K(x,y,t,y')$ defined in a (3n+1)-dimensional cylinder over S_T,

$$\Sigma_T: \quad (x,y,t)\in S_T, \quad -\infty< y_s' < \infty, \quad s = 1, \ldots, d.$$

The domain of integration here has been taken as the entire d-dimensional y'-space, although a bounded portion, or a lower dimensional subset, would have been equally acceptable. Unlike Jorgens, we do not exclude a neighborhood of the origin from the domain of integration.

Later on, S_o will denote the base, $t = 0$, of S_T. $S^{(o)}$ will be the set of points of S_T on any of the planes $y_r = 0$, $r = 1,\ldots, d$. For $\varepsilon > 0$, $S^{(\varepsilon)}$ will be an ε-neighborhood of $S^{(o)}$ in S_T.

$\mathcal{L}_T$ will denote an $(n+1)$-dimensional cylindrical drum with base $\mathcal{L}$, namely

$$\mathcal{L}_T: \quad x \in \mathcal{L}, \quad 0 \leq t \leq T .$$

By $\mathcal{L}_T^{(o)}$ will be meant the set of points of $\mathcal{L}_T$ on any of the planes $y_r = 0$, $r = 1, \ldots, d$.

The initial condition imposed is of the form

$$u(x,y,0) = \phi(x,y) . \tag{1.2}$$

The boundary condition is

$$u(\bar{x},y,t) = 0 \quad \text{when } \Sigma\, y_j N_j(\bar{x}) < 0 , \tag{1.3}$$

where $(N_j(\bar{x}))$ denotes the outward normal to $\mathcal{L}$ at $\bar{x}$, a point of the boundary not on an edge. This boundary condition can be written as

$$u(x,y,t) = 0 \quad \text{for} \quad x \in B_{r,y} , \tag{1.3'}$$

where

$$B_{r,y}: \begin{cases} |x_s| \leq C, & s \neq r \\ x_r = C_{r,y} \equiv C, \text{ if } & y_r < 0 \\ \qquad\quad \equiv -C, \text{ if } & y_r > 0 \end{cases}$$

is an appropriate one of the two faces

$$B_r^+: \quad x_r = C, \quad |x_s| \leq C \quad \text{for } s \neq r ,$$

$$B_r^-: \quad x_r = -C, \quad |x_s| \leq C \quad \text{for } s \neq r ,$$

of $\mathcal{L}$ normal to the x_r-axis. When $y_r = 0$, no boundary condition is imposed, and $B_{r,y}$ in this case is to be taken as empty. For $y_1 y_2 \cdots y_d \neq 0$, the $B_{r,y}$ $(r=1,\ldots,d)$, d mutually perpendicular planes in d-dimensional space, all intersect in one point.

By

$$B'_{r,y}:\begin{cases} |x_s| \leq C,\ x_s \neq C_{s,y} & \text{for } s \neq r \\ x_r = C & \text{if } y_r > 0 \\ \quad = -C & \text{if } y_r < 0 \\ \quad = \pm C & \text{if } y_r = 0 \end{cases}$$

we shall denote the face of $\mathcal{L}$ normal to the x_r-axis on which u goes unprescribed.

U will denote the union of the planes $y_s = 0$, $s = 1,\ldots, d$; the equation of U is $y_1 \ldots y_d = 0$.

We shall seek, depending on the precise hypotheses, either "weak" or "strong" solutions of 1.1, 1.2, 1.3.

DEFINITION 1: By a "strong" solution in S_T will be meant a bounded function $u(x,y,t)$ that, for constant x,t, is measurable with respect to y, for constant $y \notin U$ is absolutely continuous with respect to t,x in $\mathcal{L}_T - \mathcal{L}_T^{(o)}$, fulfills the initial and boundary conditions 1.2 and 1.3, and satisfies 1.1 at almost all points of S_T.

"Weak" solutions are functions that satisfy the required conditions in a certain integral sense. To arrive at an appropriate formulation, imagine u to satisfy 1.1, 1.2, and 1.3 strictly, and this and the other functions involved to admit the various operations we shall perform. We first multiply equation 1.1 by $\varphi(x,y,t)F'(u)$, where $F(u)$ and $\Phi(x,y,t)$ are continuously differentiable functions for all values of their arguments. Then we integrate over an arbitrary polyhedral domain D contained in the drum $\mathcal{L}_T$. Letting $\dot{D}$ denote the boundary of D and $(\nu_i, \nu_t)_{i=1,\ldots,d}$ the outward normal to this boundary, equation 1.1 becomes, upon integration by parts,

$$\int_{\dot{D}} \varphi\, F(u)(\nu_t + \sum_r y_r\, \nu_r)dS - \int_D F(u)(\varphi_t + \Sigma y_r\, \varphi_{x_r})dxdt$$

1.4

$$= \int_D \varphi\, F'(u)\left\{ -cu + \int Kudy' + g\right\} dxdt\ .$$

Now we specialize D to $\mathcal{L}_\tau$, $0 < \tau \leq T$, take $F(u) = u$, and impose the condition

1.5 $\varphi = 0$ on $B'_{r,y}$, $r = 1, \ldots, d.$

In view of 1.3' and 1.2, the integral over $\dot{D}$ in 1.4 thereby becomes

$$\int_{\mathcal{L}} \varphi\ udx\Big|_{t=\tau} - \int_{\mathcal{L}} \varphi\ \phi dx\ ,$$

and relation 1.4

1.6

$$\int_{\mathcal{L}} \varphi udx\Big|_{t=\tau} = \int_{\mathcal{L}} \varphi\ \phi dx + \int_{\mathcal{L}_\tau} u(\varphi_t + \Sigma y_r \varphi_{x_r})dxdt$$
$$+ \int_{\mathcal{L}_\tau} \varphi(-cu + \int Kudy' + g)dxdt\ .$$

We thus arrive at

DEFINITION 2: $u(x,y,t)$ is called a "weak" solution of 1.1, 1.2, 1.3 if relation 1.6 holds for all vectors y, all τ in the interval $(0,T)$, and all functions $\varphi(x,y,t)$ continuously differentiable in S_T that satisfy 1.5.

Appropriate function classes for u and the other quantities involved will be indicated below.

Under rather light hypotheses, e.g., hypotheses (i)-(iv) below, a bounded, absolutely continuous (in $S_T - S^{(o)}$) weak solution is a strong solution. This is so because the procedure that had led formally to 1.6 can, in the case of an absolutely continuous u, be reversed. Conversely, a strong solution is weak.

Let $V(S_T)$ denote the space of bounded functions $v(x,y,t)$ Lipschitz-continuous on S_T. Define $V'(S_T)$ as its completion under the norm

$$||v||' = \sup_{\substack{0 \le t \le T \\ \text{all } y}} \left\{ \int_{\mathcal{L}} (v(x,y,t))^2 dx \right\}^{1/2} ;$$

define $V''(S_T)$ as the completion of $V(S_T)$ under the second norm

$$||v||'' = \sup_{y} \left\{ \int_{\mathcal{L}_T} (v(x,y,t))^2 dxdt \right\}^{1/2} ;$$

let $V(S_o)$ denote the space of bounded functions $\phi(x,y)$ Lipschitz-continuous on S_o and $V'(S_o)$ the completion of this space under the norm

$$||\phi||' = \sup_{y} \left\{ \int_{\mathcal{L}} (\phi(x,y))^2 dx \right\}^{1/2} .$$

Any member of $V'(S_T)$ is a function $w(x,y,t)$ such that

$$W(y,t) = \left\{ \int_{\mathcal{L}} (w(x,y,t))^2 dx \right\}^{1/2}$$

is continuous in y,t; similar statements hold respecting the members of $V''(S_T)$ and $V'(S_o)$. To see this, let $v_k(x,y,t)$, $k = 1,2,\dots$, belonging to $V(S_T)$, approximate w in $V'(S_T)$. Thus,

$$\lim_{k,m \to \infty} \sup_{\substack{0 \le t \le T \\ \text{all } y}} \int_{\mathcal{L}} (v_k(x,y,t) - v_m(x,y,t))^2 dx = 0 .$$

This implies that the continuous functions

$$W_k(y,t) = \left\{ \int_{\mathcal{L}} (v_k(x,y,t))^2 dx \right\}^{1/2}$$

converge uniformly with respect to y,t, since by the triangle inequality we have

$$|W_k(y,t) - W_m(y,t)| \leq \left\{ \int_{\mathcal{L}} (v_k(x,y,t) - v_m(x,y,t))^2 dx \right\}^{1/2}$$

and, therefore,

$$\lim_{k,m \to \infty} \sup_{\substack{0 \leq t \leq T \\ \text{all } y}} |W_k(y,t) - W_m(y,t)| = 0 .$$

Hence, $W = \lim W_k$ is continuous, as asserted.

We shall prove the existence of weak solutions of our problem under the following hypotheses:

(i) The coefficient $c(x,y,t)$ is bounded and belongs to $V''(S_T)$.

(ii) The inhomogeneous part $g(x,y,t)$ belongs to $V''(S_T)$.

(iii) The kernel $K(x,y,t,y')$ satisfies the following conditions:

(a) $K \geq 0$.

(b) A sequence of non-negative, continuously differentiable functions $K_m(x,y,t,y')$, $m = 1,2,\ldots,$ exists such that

$$\int |K(x,y,t,y') - K_m(x,y,t,y')| dy' \leq \varepsilon_m$$

in S_T , where the ε_m are constants that approach zero as $m \to \infty$.

(c) $K(x,y,t,y') \leq K_o(y,y')$, where $K_o(y,y')$ is integrable with respect to y' and also integrable with respect to y on any sphere $|y| \leq$ const., and where

$$\int K_o(y,y')dy' \leq k_o ,$$

k_o being a constant.

(d) For all y ,

$$\lim_{y_1 \to y} \int_{\mathcal{L}_T} \left(\int |K(x,y,t,y') - K(x,y_1,t,y')| dy' \right) dxdt = 0 .$$

(iv) The initial data function $\phi(x,y)$ belongs to $V'(S_o)$.

If u is in $V'(S_T)$ and the above hypotheses are satisfied, the integrals in 1.6 will be continuous with respect to y, τ. We see this first for the integral on the left, which can be expressed as a difference,

$$\int \varphi u dx = \frac{1}{4}\int (\varphi + u)^2 dx - \frac{1}{4}\int (\varphi - u)^2 dx ,$$

of two terms already known to be continuous. Analogous considerations apply to the first and second integrals on the right and also to $\int \varphi c u dxdt$ and $\int \varphi g dxdt$. The integral

$$\int_{\mathcal{L}_\tau} dxdt \int K v dy' ,$$

in which $v = \varphi u$, is easily seen from Schwarz' inequality and hypothesis (iiic) to be continuous in τ uniformly with respect to τ, y . Hence, all that remains is to prove that

$$\int_{\mathcal{L}_\tau} \left(\int (K(x,y,t,y') - K(x,y_1,t,y'))v(x,y',t)dy' \right) dxdt$$

tends to zero as $y_1 \to y$. Fubini's theorem and Schwarz' inequality show the latter integral, in absolute value, to be

$$\leq \int dy' \left\{ \int_{\mathcal{L}_\tau} |K(x,y,t,y') - K(x,y_1,t,y')| dxdt \right\}^{1/2} \left\{ \int_{\mathcal{L}_\tau} |K(x,y,t,y') - K(x,y_1,t,y')| (v(x,y',t))^2 dxdt \right\}^{1/2}$$

$$\leq \left\{ \int_{\mathcal{L}_\tau} \left(\int |K(x,y,t,y') - K(x,y_1,t,y')| dy' \right) dxdt \right\}^{1/2}$$

$$\cdot \left\{ \int \int_{\mathcal{L}_\tau} |K(x,y,t,y') - K(x,y_1,t,y')| (v(x,y',t))^2 dxdtdy' \right\}^{1/2} ,$$

and by (iiic) to be

$$\leq (2k_o)^{1/2} ||v||'' \left\{ \int_{\mathcal{L}_T} \left(\int |K(x,y,t,y') - K(x,y_1,t,y')| dy' \right) dxdt \right\}^{1/2} .$$

This tends to zero by (iiid). Thus, all the integrals that enter 1.6 are indeed continuous in y,τ, as asserted.

THEOREM 1 (Existence of weak solutions): Under hypotheses (i)-(iv), the problem of satisfying 1.1, 1.2, 1.3 has a weak solution belonging to $V'(S_T)$.

The weak solutions we shall actually construct belong to a narrower class than that ascribed to them above, and in the narrower class they are unique, i.e., are uniquely determined by their initial data. These weak solutions, namely, all have the property, hitherto unmentioned, of satisfying the inequality

1.7 $$\int_{\mathcal{L}} (u(x,y,\tau))^2 dx \leq \int_{\mathcal{L}} (\phi(x,y))^2 dx + 2 \int_{\mathcal{L}_\tau} \left\{ -cu^2 + u \int Kudy' + gu \right\} dxdt$$

for all vectors y and all τ in the interval $(0,T)$. This inequality, like 1.6, is formally deducible from 1.4 in which we would take $D = \mathcal{L}_\tau, F(u) = u^2$ and $\varphi = 1$ to obtain

$$\int_{\dot{\mathcal{L}}_\tau} u^2(\nu_t + \Sigma_r\, y_r \nu_r)dS = 2\int_{\mathcal{L}_\tau}\left\{-cu^2 + u\int K u dy' + gu\right\} dxdt\ .$$

The left member of the latter relation, however, equals

$$\int_{\mathcal{L}}(u(x,y,\tau))^2dx - \int_{\mathcal{L}}u(x,y,0))^2dx + \Sigma_s\int_0^\tau dt \int_{B'_s} u^2\ \Sigma_r\, y_r\nu_r\, d_sx\ ,$$

d_sx denoting the element of area on B'_s, while on B'_s $\Sigma\, y_r\nu_r \geq 0$. From these remarks, inequality 1.7 easily follows, at least for strict solutions of suitably regular problems. For weak solutions as described, the inequality will be seen to carry over by closure.

THEOREM 1a: The weak solutions rendered in Theorem 1 constitute a linear class W each of whose members $u(x,y,t)$ is subject to inequality 1.7 with appropriate $\phi(x,y)$ and $g(x,y,t)$.

THEOREM 2 (Uniqueness of weak solutions): The members of W are uniquely determined by their associated ϕ's and g's: the member corresponding to $\phi = 0$ and $g = 0$, in particular, is $u = 0$.

When c, g, ϕ, and K are requisitely smooth (continuous differentiability would be enough), and, in addition, K is sufficiently attenuated near $y' = \infty$, a strong solution of the problem exists. Respecting c, g, and ϕ, the following assumptions, added to the previous, suffice:

$(i)_\alpha$ $c(x,y,t)$ and $g(x,y,t)$ are uniformly Hölder-continuous with respect to x_s, $s = 1,\ldots,d$, with Hölder exponent $\alpha(0 < \alpha \leq 1)$ and Hölder constants denoted by c_α and b_α, respectively.

$(i)_t$ $c(x,y,t)$ and $g(x,y,t)$ are uniformly Lipschitz-continuous with respect to t.

$(iv)_1$ $\phi(x,y)$ is Lipschitz-continuous with respect to x with a uniform constant denoted by ϕ_1.

(It might be expected that these conditions should be of the same kind, all Lipschitz, or all Hölder, conditions. With all Lipschitz conditions, the solution, however, still could not be proved to be more than Hölder continuous (with any exponent < 1 in an appropriate region), and with only Hölder conditions nothing could be proved at all. Therefore, the three hypotheses have been left of different kinds.)

Respecting K, four new assumptions are made:

$(v)_1$ For $t \geq 0$, $\Delta t > 0$,

$$\int \frac{|K(x,y,t+\Delta t,y') - K(x,y,t,y')|}{\Delta t}\, dy' \leq k_1 ,$$

where k_1 is a constant.

$(v)_2$ With $\Delta x_r \neq 0$, $x^{(r)} = (x_1,\ldots,x_{r-1},x_r + \Delta x_r,x_{r+1},\ldots,x_d)$,

we have

$$\int \frac{|K(x^{(r)},y,t,y') - K(x,y,t,y')|}{|\Delta x_r|^{\alpha}}\, dy' \leq k_\alpha , \qquad r = 1,\ldots,d ,$$

where k_α is a constant.

$(v)_3$ Let $y'_{(r)} = (y'_1,\ldots, y'_{r-1}, y'_{r+1},\ldots,y'_d)$. A positive λ and, for each $r = 1,\ldots,d$, a function $K_r(y'_{(r)})$, exist such that, if $|y'_r| \leq \lambda$,

$$K(x,y,t,y') \leq K_r(y'_{(r)})$$

and

$$\int K_r(y'_{(r)})\, dy'_{(r)} \leq k$$

for $r = 1,\ldots,d$, where k is a constant.

$(v)_4$ A constant K_1 exists such that

$$\int K(x,y,t,y')|y'_r|\, dy' \leq K_1\,, \quad r = 1,\ldots,d.$$

We can now state

THEOREM 3 (Existence and uniqueness of strong solutions): Under hypotheses (i)-(iv), $(i)_\alpha$, $(i)_t$, $(iv)_1$, $(v)_1$ - $(v)_4$, the problem of satisfying 1.1, 1.2, 1.3 has a strong solution, which is Hölder-continuous in $S_T - S^{(o)}$ and, for any $\varepsilon > 0$, uniformly so in $S_T - S^{(\varepsilon)}$. This solution is unique.

The paper is organized as follows:

Section 1: Statement of problem. Notation. Main results.
Weak and strong solutions are defined and the main results as to their existence and uniqueness stated.

Section 2: Uniqueness and continuous dependence. Weak solutions of the class W defined above (W includes all strong solutions) are proved to depend continuously on their initial data and thus, in particular, to be uniquely determined by their data (Theorem 2). Since the weak solutions obtained (Theorems 1 and 1a) belong to W, they are thus seen to be independent of accidental features, like selection procedures, pertaining to the construction process.

Section 3: Positivity. Monotonic dependence of solution on data, coefficient, kernel, and inhomogeneous part of equation. The solution is shown to be positive when the quantities indicated have appropriate fixed signs. All solutions, therefore, depend monotonically upon these quantities.

Section 4: Truncated problems. Proof of Theorems 1 and 1a. A special kind of problem is defined that later proves solvable by finite differences. Anticipating that the solutions of these special problems exist, we here show how they can be used to approximate the solutions of arbitrary problems. Thus, Theorems 1 and 1a are proved.

Section 5: Difference scheme notation. Some remarks. The remarks are concerned mainly with our treatment of the improper integral.

Section 6: Statement of difference equations. The difference equations are set up under assumptions deemed appropriate. They are shown to be recursive.

Section 7: Outline of convergence proof. Estimates deferred to Sections 8 and 10 prove that, as the mesh widths vary, the solutions of the difference equations are compact. This fact and uniqueness (Theorem 2 and Section 2) prove the solutions of the difference equations to converge to a weak solution of the corresponding problem.

Section 8: A bound for the solution of the difference equations and an estimate for its t-difference quotients. Two of the estimates applied in Section 7 are developed.

Section 9: Boundary behavior. The manner of the assumption of boundary data and related questions are discussed for solutions of difference equations; the results are applied in Sections 10 and 11.

Section 10: Lipschitz conditions in truncated problems. The remaining estimates required in Section 7 are given.

Section 11: Hölder conditions in untruncated problems. Proof of Theorem 3.

Estimates are obtained that have the effect, under appropriate hypotheses, of showing that a weak solution is strong.

2. Uniqueness and continuous dependence. Uniqueness and some allied properties are demonstrable under fewer restrictions as to K than those of section 1. Assumption (iii), namely, can be replaced by

$(iii)_o$ $K(x,y,t,y')$ is integrable with respect to x,t,y' on S_T. Furthermore,

$$|K(x,y,t,y')| \leq K_o(y,y') ,$$

where $K_o(y,y')$ is integrable with respect to y', and integrable with respect to y on any sphere $|y| \leq$ const., and where

$$\int K_o(y,y')dy' \leq k_o \qquad (k_o = \text{constant}) .$$

Our earlier uniqueness statement (Theorem 2) is included in

THEOREM 2.1. Under hypotheses (i) and $(iii)_o$, a function $u(x,y,t)$ belonging to $V'(S_T)$ and satisfying inequality 1.7 with $\phi = 0$, $g = 0$

is zero almost everywhere in S_T.

This fact is an obvious corollary of an estimate below for the growth of

$$U(t) = \sup_y \int_{\mathcal{L}} (u(x,y,t))^2 dx \ .$$

In stating this estimate, for convenience we suppose

$(i)_o$ c is a non-negative, bounded member of $V''(S_T)$. (Non-negativity is merely a normalization arising, for instance, as a result of a substitution $u = e^{\lambda t} v$ with sufficiently large λ.)

THEOREM 2.2 (Continuous dependence). Under hypotheses $(i)_o$, (ii), $(iii)_o$, (iv), a function $u(x,y,t)$ belonging to $V'(S_T)$ that satisfies inequality 1.7 for all y and for $0 \leq t \leq T$ also satisfies the integral inequality

$$2.1 \qquad U(t) \leq (||\phi||'^2 + ||g||''^2) e^{(2k_o + 1)t}$$

for $0 \leq t \leq T$. (The norms have been defined in Section 1.)

Proof: From 1.7, since $c \geq 0$, we have

$$2.2 \qquad \int_{\mathcal{L}} (u(x,y,\tau))^2 dx \leq ||\phi||'^2 + 2 \int_{\mathcal{L}_\tau} \left\{ u \int K u dy' + gu \right\} dx dt$$

for all vectors y and all τ in the interval $(0,T)$. By Schwarz' inequality and $(iii)_o$,

$$\left(\int K u dy' \right)^2 \leq \left(\int |K| dy' \right) \left(\int |K| u^2 dy' \right)$$

$$\leq k_o \int K_o(y,y') \, (u(x,y',t))^2 dy' \ ;$$

hence, by Fubini's theorem,

$$2.3 \qquad \int_{\mathcal{L}} \left(\int K u dy' \right)^2 dx \leq k_o \int_{\mathcal{L}} dx \int K_o(y,y') \, (u(x,y',t))^2 \, dy'$$
$$= k_o \int K_o(y,y') dy' \int_{\mathcal{L}} (u(x,y',t))^2 \, dx$$
$$\leq k_o^2 U(t).$$

Consequently,

$$\left| \int_{\mathcal{L}_\tau} u \left(\int K u dy' \right) dx dt \right| \leq \left\{ \int_{\mathcal{L}_\tau} u^2 dx dt \int_{\mathcal{L}_\tau} \left(\int K u dy' \right)^2 dx dt \right\}^{1/2}$$
$$\leq k_o \int_0^\tau U(t) dt \, .$$

This, inequality 2.2 , and Schwarz' inequality applied to $\int gudxdt$ enable us to deduce from relation 2.2 the inequality

$$U(\tau) \leq ||\phi||'^2 + 2||g||'' \left(\int_0^\tau U(t) dt \right)^{1/2} + 2k_o \int_0^\tau U(t) dt$$
$$\leq ||\phi||'^2 + ||g||''^2 + (2k_o + 1) \int_0^\tau U(t) dt \, ,$$

valid for $0 \leq \tau \leq T$. It is easily seen that, for $0 \leq t \leq T$, $U(t) \leq V(t)$, where $V(t)$ is determined by the conditions

$$V'(t) = (2k_o + 1) V(t)$$
$$V(0) = ||\phi||'^2 + ||g||''^2 \, .$$

Thus, estimate 2.1 is justified.

3. Positivity. Monotonic dependence of solution on data, coefficient, kernel, and inhomogeneous part of equation. When the coefficient, the inhomogeneous part, and the data are positive, the solution will be positive, too. This is more precisely stated in

THEOREM 3.1. Under hypotheses (i)-(iv), let $u(x,y,t)$ denote the weak solution belonging to class W of which we are assured in Theorems 1 and 1a. If

$$c \geq 0, \quad g \geq 0, \quad \phi \geq 0,$$

then

$$u \geq 0$$

at almost all points of S_T.

A significant consequence of positivity is that u depends monotonically on $-c$, g, K, and ϕ, as we assert in

THEOREM 3.2. Consider two problems of the form specified in 1.1, 1.2, and 1.3, each satisfying hypotheses (i)-(iv). Distinguishing corresponding quantities in the two problems by the subscript 1 or 2, assume $c_1 \geq 0$ and

$$c_2 \geq c_1, \quad g_1 \geq g_2, \quad K_1 \geq K_2, \quad \phi_1 \geq \phi_2;$$

denote by u_1 and u_2 the weak solutions, in W, of these respective problems as provided by Theorem 1. At almost all points of S_T,

$$u_1 \geq u_2.$$

Theorem 3.2 is proved by noting that the difference $v = u_1 - u_2$ initially is non-negative and is a weak solution of the equation

$$v_t + \Sigma y_j v_{x_j} + c_1 v = \int K_1 v dy' + \left((c_2 - c_1)u_2 + \int (K_1 - K_2) u_2 dy' + g_1 - g_2\right) .$$

The expression in parentheses being non-negative, Theorem 3.1 applies.

Theorem 3.1 need be proved but for bounded, Lipschitz-continuous solutions since, as appears from Sections 2, 4, and 7, the (weak) solution of an arbitrary problem can be approximated by solutions, which are bounded and Lipschitz-continuous, of related, approximating, "truncated" problems. We shall therefore assume that u is bounded and Lipschitz-continuous. (Lipschitz-continuity with respect to x,t for almost all y, as is the case for a strong solution in the absence of truncation, would work equally well.) Relation 1.4 holds for any Lipschitz-continuous solution, and we may make the following substitutions:

$$F(u) = u , \quad u = e^{2k_o\delta t} v, \quad \varphi = e^{-2k_o\delta t} ,$$

where δ is an arbitrary number > 1. Thereby, 1.4 becomes

3.1 $$\int_D v(v_t + \Sigma y_r v_r) dS = \int_D \left\{-(c+2k_o\delta)v + \int K v dy' + g e^{-2k_o\delta t}\right\} dxdt ,$$

a relation for the new dependent variable v. Obviously, it suffices to prove

3.2 $$v \geq 0 \quad \text{in } S.$$

Imagine that, to the contrary, for some positive value of T we have

3.3 $$m = \inf_{\substack{x \in \mathfrak{X} \\ \text{all } y \\ 0 \leq t \leq T}} v(x,y,t) < 0 .$$

In this case, a vector y^* exists such that

$$3.4 \qquad m^* = \min_{\substack{x \,\varepsilon\, \mathfrak{X} \\ 0 \le t \le T}} v(x,y^*,t) < m/\delta \ .$$

Let the minimum m^* be realized at a point P^*: (x^*,t^*), $0 < t^* \le T$, i.e.,

$$3.5 \qquad v(x^*,y^*,t^*) = m^* \ ,$$

and assume t^* the least value of t at which this minimum occurs. Thus,

$$3.6 \qquad v(x,y^*,t) > m^* \quad \text{for} \quad 0 \le t < t^* \ , \quad x \,\varepsilon\, \mathfrak{X} \ .$$

By relations 3.3 and 3.4 ,

$$\int K(x,y,t,y') \ v(x,y',t)dy' \ge k_o m > k_o \delta \ m^* \ .$$

Hence, and because $g \ge 0$, equation 3.1 implies

$$\int_{\dot{D}} v(v_t + \Sigma y_r v_r)dS > - \int_D cvdxdt - k_o\delta \int_D (2v - m^*)dxdt \ .$$

If N is a neighborhood of P^* such that $v(x,y^*,t) < m^*/2$ in $\mathfrak{X}_T \cap N$, we thus have, since $c \ge 0$,

$$3.7 \qquad \int_{\dot{D}} v(v_t + \Sigma \ y_r v_r)dS > 0$$

for $D \subset \mathfrak{X}_T \cap N$. This, however, is incompatible with the minimality of m^* expressed in 3.5 and 3.6 , which we see as follows.

For $t_o > t^*$ and $h > 0$, consider the truncated (d+1)-dimensional pyramid

$$Pyr_{t_o,h}: \quad \begin{array}{l} |x_j - x_j^*| \le |y_j^*|(t_o - t) \ , \quad j = 1,\dots,d, \\ t^* - 3h \le t \le t^* \end{array}$$

with altitude 3h. The apex of the pyramid is at the point (x^*, t_o). We choose h and $t_o - t^*$ so small that

3.8 $\quad \mathrm{Pyr}_{t_o,h} \subset N$.

Evidently, 3.7 then holds whenever $D \subset R$, where

$$R = \mathrm{Pyr}_{t_o,h} \cap \mathcal{L}_T \ .$$

The boundary of R consists of a sloping portion belonging to the pyramid and also, possibly, of a "vertical" piece, or pieces, over the boundary of $\mathcal{L}$. (At a point $(x_1, \ldots, x_d, t)$ of a "vertical" piece of the boundary of R, some $x_s = \pm C$.)

With $0 < \eta < h$, let

$$P^{(\eta)}: \quad (x^{(\eta)}, \quad t^{(\eta)})$$

denote an interior point of R not farther from P* than the distance η. Assuming $t^* - h < t^{(\eta)} \leq t^*$, consider the parallelepiped

$$Q_{\varepsilon,\eta}: \quad \begin{array}{l} |x_i - x_i^{(\eta)} - y_i^*(t - t^{(\eta)})| \leq \varepsilon/2 , \quad i = 1, \ldots, d , \\ 0 \leq t^{(\eta)} - t \leq h . \end{array}$$

For sufficiently small ε,

3.9 $\quad Q_{\varepsilon,\eta} \subset \mathrm{Pyr}_{t_o,h}$.

The two horizontal faces

$$F_1: \quad \begin{array}{l} |x_i - x_i^{(\eta)} + y_i^* h| \leq \varepsilon/2 , \quad i = 1, \ldots, d , \\ t = t^{(\eta)} - h \end{array}$$

and

$$F_2: \quad \begin{array}{l} x_i - x_i^{(\eta)} \leq \varepsilon/2 , \quad i = 1, \ldots, d , \\ t = t^{(\eta)} \end{array}$$

of the parallelepiped are each of area ε^d . The non-horizontal $(d-1)$-dimensional faces are parts of the planes

$$x_i = x_i^{(\eta)} + y_i^*(t - t^{(\eta)}) \pm \varepsilon/2 ;$$

for the normal $\nu = (\nu_1, \ldots, \nu_d, \nu_t)$ to one of these faces we note that

$$3.10 \qquad \nu_t + \sum_{i=1}^{d} y_i^* \nu_i = 0 .$$

Knowing P^* , we can select h such that, for any selection of $P^{(\eta)}$ and all sufficiently small ε ,

$$3.11 \qquad Q_{\varepsilon,\eta} \subset R .$$

If x^* is interior to $\mathfrak{X}$, this is evident from 3.9 , for h in this case can be so reduced that the space component x of any point (x,t) of $Pyr_{t_o,h}$ also will be interior to $\mathfrak{X}$. If, on the other hand, x^* is on the boundary of $\mathfrak{X}$, we see from 1.3 and 3.5 that, at x^* , the vector y^* points outward from $\mathfrak{X}$. Hence, the characteristic — — — the straight line with slopes $dx_i/dt = y_i^*$, $i = 1, \ldots, d$ — — — through P^*, positively oriented in the sense of increasing t , points outward from R : the segment of the characteristic for which $t^*-h \leq t \leq t^*$, in particular, is contained within R . Corresponding segments of neighboring (parallel) characteristics also will be contained within R . $Q_{\varepsilon,\eta}$, however, is generated by such segments. Hence, $P^{(\eta)}$ being selected, $Q_{\varepsilon,\eta}$ will be contained within R, as asserted, if ε is sufficiently small.

Because of 3.11 and 3.8 , we can choose $D = Q_{\varepsilon,\eta}$ in 3.7 . In view of 3.10, inequality 3.7 then reduces to

3.12 $$\int_{F_2} v\,dx - \int_{F_1} v\,dx > 0 .$$

Dividing this by ε^d and letting $\varepsilon \to 0$ proves

$$v(P^{(\eta)}) - v(P^{(\eta)**}) \geq 0 ,$$

where $P^{(\eta)**}$ is the point with the coordinates $x_i^{(\eta)} - y_i^* h$, $t^{(\eta)} - h$. Now letting $\eta \to 0$ proves

$$v(P^*) - v(P^{**}) \geq 0 ,$$

where P^{**} is a point on the plane $t = t^* - h$. This contradicts the minimality of m^* , i.e., contradicts 3.5 and 3.6 , and in view of the boundary condition we conclude that $m^* = 0$. This means statement 3.2 holds, all we needed to prove.

4. Truncated problems. Proof of Theorems 1 and 1a. A problem of the type described in section 1 will be called "truncated" if $g(x,y,t)$, like $K(x,y,t,y')$, is non-negative and both are zero when y is in a neighborhood of the union U of the planes $y_s = 0$, $s = 1, \ldots, d$. Truncated problems play an essential role in our proof of Theorems 1 and 1a, for their solutions, properly combined, approximate the solution of any problem, while they themselves are solvable by finite difference methods. Their solvability will be discussed in section 7, their approximative capability here: the effect of the combined discussion is to prove Theorems 1 and 1a.

Our present aim, stated more precisely, is to show that the weak solution of an arbitrary problem can be approximated in $V'(S_T)$ by the solutions of appropriate truncated problems with, say, continuously differentiable coefficient, kernel, data and inhomogeneous part. We anticipate the fact, to be proved in Section 7, that such truncated problems have Lipschitz-continuous strong solutions. Three steps are involved.

The first step is to understand that, under any stipulated boundary and initial conditions, equation 1.1 is solvable for any g satisfying hypothesis (ii) if solvable for any such non-negative g. Any g subject to (ii) can be represented, in fact, as a difference

$$g = g_1 - g_2 \qquad (g_1, g_2 \geq 0)$$

of non-negative functions also subject to (ii). If u_1 and u_2 are weak solutions of the respective relations

$$Lu_1 = g_1 + \int Ku_1 dy' , \quad Lu_2 = g_2 + \int Ku_2 dy' ,$$

where $Lv = v_t + \sum_r y_r v_{x_r} + cv$, then $u = u_1 - u_2$ evidently is a weak solution of the equation

$$Lu = g + \int Kudy' .$$

Thus, it suffices to be able to solve 1.1 for non-negative g, as asserted. In what follows, we shall generally assume $g \geq 0$.

The second step is concerned with problem 1.1, 1.2, 1.3 when $g \geq 0$ and c, g, K, and ϕ are all continuously differentiable besides satisfying hypotheses (i) to (iv); for convenience, $(i)_o$ (Section 2) also is assumed. The (weak) solution of such a problem will be exhibited as the limit of solutions of appropriate truncated problems. It will belong to W. More, if u and v are the solutions of two such problems, say with

$$\begin{cases} u_t + \Sigma y_r u_{x_r} + c_1 u = g_1 + \int K_1 u dy' \\ u(x,y,0) = \phi_1(x,y) , \end{cases}$$

$$\begin{cases} v_t + \Sigma y_r v_{x_r} + c_2 v = g_2 + \int K_2 v dy' \\ v(x,y,0) = \phi_2(x,y) , \end{cases}$$

the difference $u-v$ will be shown to satisfy relation 1.7 with $c = c_1$, $K = K_1$, $g = g_1 - g_2 + (c_2 - c_1)v + \int (K_1 - K_2) v \, dy'$, $\phi = \phi_1 - \phi_2$; by Theorem 2.2, it will then follow that

$$4.1 \quad ||u-v||'^2 \leq C(||\phi_1-\phi_2||'^2 + ||g_1-g_2||''^2 + ||(c_2-c_1)v||''^2 + (\max_{S_T} \int |K_1 - K_2| dy')||v||'^2)$$

where C is a constant. (An estimation is used like that for 2.3.) This step, the main task of the present section, will be carried out presently.

The third step is to perceive that the solution of a general problem, one satisfying hypotheses (i) to (iv), can be approximated by solutions of problems of the type considered in the second step. The latter solutions belonging to W, their limit, the solution of the given general problem, too belongs to W: Theorems 1 and 1a are thus proved. We shall carry out the third step now, assuming the second to be valid, and then return to justify the latter. We make assumption $(i)_o$ for convenience, in place of (i), and take $g \geq 0$, as from the first step we may. Then we approximate ϕ in $V'(S_o)$ by a sequence of continuously differentiable functions

$\phi_k(x,y)$, $k = 1,2, \ldots, g$ in $V''(S_T)$ by a sequence of non-negative, continuously differentiable functions $g_k(x,y,t)$, $k = 1,2,\ldots$, and c in $V''(S_T)$ by a bounded sequence of non-negative, continuously differentiable functions $c_k(x,y,t)$, $k = 1,2,\ldots$; K we approximate by the K_k of hypothesis (iiib). It follows from step two that, for each k , the problem corresponding to ϕ_k, c_k, g_k, K_k has a solution u_k with uniformly bounded norm $||u_k||'$ and that, for all k and m,

$$||u_k - u_m||'^2 \leq C(||\phi_k - \phi_m||'^2 + ||g_k - g_m||''^2 + ||(c_k - c_m)u_m||''^2$$

$$+ (\varepsilon_k + \varepsilon_m)||u_m||'^2) .$$

Each term on the right tends to zero, however, as $k,m \to \infty$, because of (iiib), other stipulated approximation properties, and the existence of uniform bounds for the c_k and the $||u_k||'$. Hence, the u_k converge in $V'(S_T)$, and, as is easily seen, the limit u is a weak solution of the given problem belonging to W . Theorems 1 and 1a thus hang on the validity of the second step.

In justifying the second step, we shall actually do more. By $V^+(S_T)$ will be meant the class of functions w , defined on S_T , each of which is the limit of a monotonically increasing sequence of functions belonging to $V(S_T)$. Analogously, by $V^+(S_o)$ will be meant the class of functions defined in S_o each of which is the limit of a monotonically increasing sequence of functions belonging to $V(S_o)$. We replace the earlier hypotheses by the following:

$(i)^+$ $c(x,y,t)$ is bounded, and $-c$ belongs to $V^+(S_T)$.

$(ii)^+$ $g(x,y,t)$ is non-negative and belongs to $V^+(S_T)$.

$(iii)^+$ (a) $K \geq 0$.

(b) A non-decreasing sequence of non-negative, continuously differentiable functions $K_m(x,y,t,y')$, $m = 1,2,\ldots,$ exists such that

$$K(x,y,t,y') = \lim_{m \to \infty} K_m(x,y,t,y')$$

at all points of Σ_T .

(c) the same as (iiic) .

$(iv)^+$ ϕ belongs to $V^+(S_o)$.

These assumptions obviously are satisfied when $g \geq 0$, ϕ, c, g, and K are continuously differentiable, and (iiia,c) hold. Hence, the second step in the argument above will be justified when we prove

THEOREM 4.1. Under hypotheses $(i)^+ - (iv)^+$, a weak solution of class W of 1.1, 1.2, 1.3 is obtainable as the monotone limit of solutions of appropriate truncated problems. If u and v are the solutions of different problems, each satisfying hypotheses $(i)^+ - (iv)^+$, u-v is subject to an inequality of the form 4.1 .

The last statement is an obvious corollary of the first, the inequality in question being obtainable as the limit of analogous inequalities for the (Lipschitz-continuous) solutions of the approximating truncated problems. The first statement depends on a suitable procedure of setting up truncated problems to approximate the given one.

Let $\varphi(s)$ be a monotonic, infinitely differentiable function on the positive semi-axis $s \geq 0$ such that

$$\varphi(s) = 0 \quad \text{for} \quad 0 \leq s \leq 1/2$$
$$\quad = 1 \quad \text{for} \quad s \geq 1 \ .$$

For $m > 1$, the function

$$\varphi_m(s) = \varphi(2s^{1/m})$$

thus is monotonic and infinitely differentiable and satisfies the conditions

$$\varphi_m(s) = 0 \quad \text{for} \quad s \leq (1/4)^m$$
$$\quad = 1 \quad \text{for} \quad s \geq (1/2)^m \ .$$

Moreover, for $s > 0$

$$\varphi_m(s) \nearrow 1 \qquad \text{as} \quad m \nearrow \infty \ .$$

If $p(y)$ is a non-negative function of the vector $y = (y_1, \ldots, y_d)$, we define a "truncation" of p as

$$p_m(y) = \prod_{j=1}^{d} \varphi_m(|y_j|) \cdot p(y) \ .$$

For $y \notin U$, a truncation evidently tends monotonely to its "original":

$$p_m(y) \nearrow p(y) \qquad \text{as} \quad m \to \infty \qquad \text{if} \quad y \notin U \ .$$

Let us now consider a problem satisfying hypotheses $(i)^+ - (iv)^+$. We shall first approximate $\phi(x,y)$, $-c(x,y,t)$, $g(x,y,t)$, and $K(x,y,t,y')$ by non-decreasing sequences $\phi_k(x,y)$, $-c_k(x,y,t)$, $g_k(x,y,t)$, and $K_k(x,y,t,y')$ $(k = 1,2,\ldots)$, respectively, of continuously differentiable functions. We then truncate the g_k and K_k with respect to y. The truncated functions, denoted by

$$g_{km}(x,y,t), \quad m = 1,2, \ldots,$$

$$K_{km}(x,y,t,y'), \ m = 1,2,\ldots,$$

for fixed k increase monotonely with m. They define, for each k and m, what we have called a truncated problem. Any such truncated problem, however, as will be seen below (section 7), can be solved (by finite difference methods); its solution u_{km} is bounded uniformly with respect to k and m (Theorem 8.1), is Lipschitz-continuous and thus satisfies appropriate relations of the form 1.6 and 1.7, and increases monotonely with m and with k (Theorem 3.2). Hence, the limit

$$u = \lim_{k\to\infty} \lim_{m\to\infty} u_{km}$$

exists and satisfies 1.6 and 1.7 , i.e., is a weak solution of class W of the original problem. This proves Theorem 4.1.

The remainder of the paper will be concerned with suitably smooth, truncated problems.

5. Difference scheme notation. Some remarks. Let $\xi > 0$, $\eta > 0$, $\tau > 0$ with $I = C/\xi$ an integer, and set

$$\theta = \tau/\xi \ .$$

Let L be a positive integral multiple of η. By i and j we shall mean multi-indices

$$i = (i_1, \ldots, i_d), \quad -I \leq i_s \leq I, \quad s = 1, \ldots, d,$$
$$j = (j_1, \ldots, j_d), \quad |j_s| < L, \qquad s = 1, \ldots, d,$$

with integral components, and by $\xi i = (\xi i_1, \ldots, \xi i_d)$ and

$\eta_j = (\eta j_1, \ldots, \eta j_d)$ corresponding points of $\mathcal{L}$ and of y-space, respectively. Symbols such as i', $\xi\, i'$, j', $\eta\, j'$ will be similarly used. For any multi-index, such as i, we shall write

$$|i| = \max_r |i_r| .$$

The set of all multi-indices i such that, for some $s = 1, \ldots, d$,

$$i_s = I , \quad |i_r| \leq I \quad \text{for } r \neq s ,$$

will be called $\mathcal{J}_s^+$, and the set such that, for some s,

$$i_s = -I , \quad |i_r| \leq I \quad \text{for } r \neq s ,$$

$\mathcal{J}_s^-$. These sets correspond to B_s^+ and B_s^-, respectively. The sets

$$\begin{aligned} \mathcal{J}_{s,j} &= \mathcal{J}_s^+ \quad \text{for } j_s < 0 \\ &= \mathcal{J}_s^- \quad \text{for } j_s > 0 \\ &= \text{empty set for } j_s = 0 \end{aligned}$$

correspond to $B_{s,y}$ and the sets

$$\begin{aligned} \mathcal{J}'_{s,j} &= \mathcal{J}_s^- - \bigcup_{r=1}^{d} \mathcal{J}_{r,j} \quad \text{for } j_s < 0 \\ &= \mathcal{J}_s^+ - \bigcup_{r=1}^{d} \mathcal{J}_{r,j} \quad \text{for } j_s > 0 \\ &= \mathcal{J}_s^- + \mathcal{J}_s^+ \quad \text{for } j_s = 0 \end{aligned}$$

to $B'_{s,y}$. (Recall that u is prescribed zero on $B_{s,y}$ and is unprescribed on $B'_{s,y}$.)

Let $S_L^{\xi,\eta,\tau}$ denote the set of lattice points $(\xi i, \eta j, n\tau)$ in xyt-space, where i and j are multi-indices as above, and n is a non-negative integer. Our aim in the difference scheme below will be to approximate the solution $u(x,y,t)$ of a given, truncated problem by functions $u_L^{\xi,\eta,\tau}(x,y,t)$ defined on $S_L^{\xi,\eta,\tau}$. When ξ, η, τ, and L are regarded as fixed, for brevity we write

$$u_L^{\xi,\eta,\tau}(\xi i, \eta j, n\tau) = u_{ij}^n ,$$

and for any function $\psi(x,y,t)$, unless otherwise specified,

$$\psi(\xi i, \eta j, n\tau) = \psi_{ij}^n , \quad \psi_{x_s}(\xi i, \eta j, n\tau) = \psi_{x_s,ij}^n ,$$

etc. In the case of the kernel K, we write

$$K_{ijj'}^n = \tau^{-1}\xi^{-d}\eta^{-2d} \int_{\mathcal{P}_{ijj'}^n} K(x,y,t,y')\, dy'dxdtdy ,$$

the domain of integration being the parallelepiped

$$\mathcal{P}_{ijj'}^n : \quad \begin{array}{ll} \xi i_s \leq x_s < \xi(i_s + 1) , & s = 1, \ldots, d , \\ \eta j_s \leq y_s < \eta(j_s + 1) , & \text{"} \\ \eta j_s' \leq y_s' < \eta(j_s' + 1) , & \text{"} \\ n\tau \leq t < \tau(n + 1) . & \end{array}$$

(In this definition, it is presumed that K is defined for $-I \leq x_s \leq I+\xi$. We might, for instance, take $K = 0$ for any $x_s > I$.) When all the definitions of this section are concluded, we shall come back to note certain consequences of this one.

Let X_r be the translation operator whose effect on the multi-index i is to increase its r-th component by one while not changing the other components:

$$X_r(i) \equiv X_r(i_1, \ldots, i_{r-1}, i_r, i_{r+1}, \ldots, i_d)$$

$$= (i_1, \ldots, i_{r-1}, i_r + 1, i_{r+1}, \ldots, i_d) .$$

The effect on a function indexed with i, say ψ^n_{ij}, is defined by

$$X_r \psi^n_{ij} = \psi^n_{X_r(i),j} .$$

The inverse of this operator, denoted by X_r^{-1}, reduces i_r by one while leaving all other indices the same. The operator

$$T_{r,j} = X_r \quad \text{for} \quad j_r < 0$$

$$= X_r^{-1} \quad \text{for} \quad j_r > 0$$

has the effect of moving i nearer $\vartheta_{r,j}$.

As analogous to X_r, let Y_r denote the translation operator whose effect is to increase j_r by one while not affecting other indices, and Y_r^{-1} its inverse. In terms of these, conventional x- and y-difference operators are introduced as

5.1 $$\delta_r = X_r - 1, \quad \varepsilon_r = Y_r - 1, \quad r = 1, \ldots, d ,$$

and

$\overline{5.1}$ $$\bar{\delta}_r = 1 - X_r^{-1} , \quad \bar{\varepsilon}_r = 1 - Y_r^{-1} , \quad r = 1, \ldots, d ,$$

but we also shall use

5.2 $\delta_{r,j} = 1 - X_r^{-1}$ when $j_r > 0$,

$= X_r - 1$ when $j_r < 0$,

and

$\overline{5.2}$ $\bar{\delta}_{r,j} = X_r - 1$ when $j_r > 0$,

$= 1 - X_r^{-1}$ when $j_r < 0$.

We note that

5.3 $j_r \, \delta_{r,j} = |j_r| \, (1 - T_{r,j})$;

this property will turn out to be a significant one.

We now return, as promised, to our kernel approximation. Anticipating hypothesis $(iii)_1$ in section 6, assume $K(x,y,t,y')$ to be continuous at almost every point of Σ_T . Note at the same time that the step function

$$K_L^{\xi,\eta,\tau}(x,y,t,y') = K^n_{ijj'} \quad \text{in } \varphi^n_{ijj'} \quad \text{for } |j| \, , \, |j'| \leq L$$

$$= 0 \quad \text{for } |j| > L \text{ or } |j'| > L$$

tends to $K(x,y,t,y')$ in Σ_T at each of its points of continuity, as $\xi, \eta, \tau \to 0, \; L \to \infty$. Then

5.4 $\lim\limits_{\substack{\xi,\eta,\tau \to 0 \\ L \to \infty}} K_L^{\xi,\eta,\tau}(x,y,t,y') = K(x,y,t,y')$ almost everywhere in Σ_T .

Furthermore,

5.5 $\int K_L^{\xi,\eta,\tau}(x,y,t,y') \, dy' \leq \int K(x,y,t,y') \, dy' \leq k_o$.

These facts suggest that we attempt to approximate the improper integral in equation 1.1 by sums (these are essentially Riemann sums) of the form

$$\eta^d \sum_{|j'| \leq L} K^n_{ijj'} \, u^n_{ij'} \quad .$$

The attempt succeeds when the step functions

$$u_L^{\xi,\eta,\tau}(x,y,t) = u^n_{ij} \quad \text{for} \begin{cases} n\,\tau \leq t < (n+1)\tau \\ \xi\, i_s \leq x_s < (i_s + 1)\xi \qquad (s=1,\ldots,d) \\ \eta\, j_s \leq y_s < (j_s + 1)\eta \end{cases}$$

are uniformly bounded and tend to a limit $u(x,y,t)$ at almost all points (x,y,t) as $L \to \infty$ and ξ, η, $\tau \to 0$, the approach of each parameter to its limit being through suitable values. To be more specific, let ξ_k, η_k, τ_k, L_k, $k = 1,2,\ldots$, be sequences of values of ξ, η, τ, L, respectively, such that the functions

$$u_k(x,y,t) \equiv u_{L_k}^{\xi_k,\eta_k,\tau_k}(x,y,t) \; ,$$

as stipulated, are uniformly bounded and tend to $u(x,y,t)$ almost everywhere as $k \to \infty$. In terms of

$$K_k(x,y,t,y') \equiv K_{L_k}^{\xi_k,\eta_k,\tau_k}(x,y,t,y') \quad ,$$

the Riemann sums considered can be written as

$$\int K_k(x,y,t,y') \, u_k(x,y,t,y') \, dy' \; .$$

We shall need to prove the existence of the limit of these sums as $k \to \infty$ only in a weak sense to be described below. Let

$$Q_{ab}: \quad a_s \leq y_s \leq b_s , \quad s = 1, \ldots, d ,$$

be a parallelepiped in y-space. Let $v(x,t)$ be a continuous function which vanishes for $x \notin \mathcal{C}$ and for $t \geq T$, where T is an arbitrary positive constant. We shall need to know only that, under the hypotheses explained, for any v as described and any numbers a and b, we have

5.6 $$\int_{Q_{ab}} dy \int v dx dt \int (K_k u_k - Ku) dy' \to 0 ,$$

as $k \to \infty$.

To see this, we write the integrand as

$$K(u_k - u) + u_k(K_k - K)$$

and express the above integral as the sum of two, the first J_{1k} corresponding to the first summand above and the second J_{2k} corresponding to the second summand. The boundedness of $|u_k|$, the convergence of u_k and K_k, and 5.5 show that J_{1k} and J_{2k} both $\to 0$ as $k \to \infty$. Thus, contention 5.6 is proved.

We also remark that, from the definition of $K^n_{ijj'}$ and hypothesis (iii),

5.7 $$\eta^d \sum_{j'} K^n_{ijj'} \leq k_o$$

(this is the same statement as was made in 5.5 .) When assumption $(iii)_1$ below is made concerning difference quotients for K, the following analogous statement can be made about $K^n_{ijj'}$: if δ stands for δ_r/ξ ,

ε_r/η , or the analogous differencing operator with respect to t ,

5.8 $$\eta^d \sum_{j'} |\delta\, K^n_{ijj'}| \leq k_1 \ .$$

6. <u>Statement of difference equations.</u> The difference equations set up below will be justified only for truncated problems in which, besides assumptions $(i)_o$, (ii), (iii), (iv), $(v)_4$ of sections 1 and 2, the following additional hypotheses also are satisfied:

$(i)_1$ $c(x,y,t)$ and $g(x,y,t)$ are Lipschitz-continuous with respect to x,y,t with uniform constants we shall denote by c_1 and b_1 , respectively. Furthermore, $0 \leq c \leq c_o$ and $|g| \leq b_o$, where c_o and b_o are constants.

$(iii)_1$ $K(x,y,t,y')$ is continuous at almost every point of Σ_T . If $DK(x,y,t,y')$ symbolizes a difference quotient of $K(x,y,t,y')$ with respect to any of the $2d + 1$ arguments x_r, y_s, t, $r,s = 1, \ldots, d$, e.g.,

$$\frac{K(x,y,t+\Delta t,y') - K(x,y,t,y')}{\Delta t} \qquad (\Delta t > 0) ,$$

we have for a suitable constant k_1

$$\int |DK(x,y,t,y')|dy' \leq k_1 \ .$$

$(iv)_1$ $\phi(x,y)$ is Lipschitz-continuous with respect to x with a uniform constant denoted by ϕ_1. Also $|\phi| \leq \phi_o$, where ϕ_o is a constant.

$(iv)_y$ $\phi(x,y)$ is uniformly Lipschitz-continuous with respect to y .

We now describe our difference scheme. Corresponding to 1.2 is the initial condition

6.1 $$u^o_{ij} = \phi(\xi\, i, \eta j) ,$$

and corresponding to 1.3 the boundary condition

6.2 $$u_{ij}^{n} = 0 \quad \text{on } \mathcal{J}_{s,j}, \quad s = 1, \ldots, d .$$

Corresponding to the integro-differential equation 1.1 we set up the difference equations

6.3 $$\frac{1}{\tau}(u_{ij}^{n+1} - u_{ij}^{n}) + \sum_{s} \eta\, j_s \frac{1}{\xi}\, \delta_{s,j} u_{ij}^{n+1} + c_{ij}^{n} u_{ij}^{n} = g_{ij}^{n} + \eta^{d} \sum_{|j'| \leq L} K_{ijj'}^{n} u_{ij'}^{n}$$

for the following values of the indices:

6.4 $$n = 0, 1, \ldots; \quad |j| \leq L ; \quad |i| < I \ \underline{\text{or}}\ \ i \in \mathcal{J}'_{s,j} .$$

This scheme is recursive: for each choice of n and j, the values of u_{ij}^{n+1}, appropriately ordered, are successively obtainable from previously calculated quantities. This fact will be more apparent from a rewriting of equations 6.3 in terms of the $T_{s,j}$. We simply apply 5.3 in 6.3. Then, rearranging, we have, as equivalent to 6.3,

6.5 $$\left(1 + \theta \sum_{s} |j_s| \eta\right) u_{ij}^{n+1} = (1 - \tau c_{ij}^{n}) u_{ij}^{n} + \theta \sum_{s} |j_s| \eta T_{s,j} u_{ij}^{n+1} + \tau\, g_{ij}^{n} + \tau\, \eta^{d} \sum_{|j'| \leq L} K_{ijj'}^{n} u_{ij'}^{n} .$$

Assuming the u_{ij}^{n} to be known for fixed n and all i and j, we shall see from this how the u_{ij}^{n+1} are determined. The determination is immediate in the case $j = 0$. In the contrary case, $j \neq 0$, u_{ij}^{n+1} is expressed in 6.5

in terms of quantities indexed with n, presumed known, and quantities of the type $u^{n+1}_{i'j}$ for which i' falls one step nearer some $\mathcal{J}_{s,j}$ than i (the $\mathcal{J}_{s,j}$, we recall, carry boundary data). To be more exact, consider a particular $j \neq 0$. The intersection

$$\bigcap_{j_s \neq 0} \mathcal{J}_{s,j}$$

of all the $\mathcal{J}_{s,j}$ for non-vanishing j_s is not empty; let i^*_j denote any index belonging to this intersection, and set

$$i'_j = \Big(\prod_{j_s \neq 0} T^{-1}_{s,j}\Big) i^*_j ,$$

the product in the right member to include every operator $T^{-1}_{s,j}$ for which $j_s \neq 0$. As i^*_j varies over all possible positions and k_s, $s = 1,\dots,d$, over all needful, non-negative integers, the index

$$\Big(\prod_{j_s \neq 0} T^{-k_s}_{s,j}\Big) i'_j$$

varies over all the lattice points i at which u^{n+1}_{ij} is not already prescribed in advance. Thus it suffices to show that, for any i^*_j, the quantities

$$\Big(\prod_{j_s \neq 0} T^{-k_s}_{s,j}\Big) u^{n+1}_{i'_j j} \tag{6.6}$$

can be recursively determined. First, $u^{n+1}_{i'_j j} \equiv \Big(\prod_{j_s \neq 0} T^{-1}_{s,j}\Big) u^{n+1}_{i^*_j j}$ can be calculated from 6.3, u^{n+1}_{ij} being known on $\mathcal{J}_{s,j}$ when $j_s \neq 0$. From this result, $T^{-1}_{s,j} u^{n+1}_{i'_j j}$, for $j_s \neq 0$, similarly can be found

Induction would show that all the quantities 6.6, in suitable order, can be determined by appropriately continuing this procedure. Hence, the scheme 6.1-3 is recursive, as asserted.

For future reference, we note that equations 6.3 also may be written as

6.7
$$(1 + \theta \sum_s |j_s|\eta) u_{ij}^{n+1} = (1 - \tau c_{ij}^n) u_{ij}^n + \theta \sum_s B_{sj} X_s u_{ij}^{n+1}$$
$$+ \theta \sum_s C_{sj} X_s^{-1} u_{ij}^{n+1} + \tau g_{ij}^n + \tau \eta^d \sum_{|j'| \leq L} K_{ijj'}^n u_{ij}^n ,$$

where, for $j_s \geq 0$,

6.8a $\quad B_{sj} = 0, \quad C_{sj} = j_s \eta$,

and, for $j_s < 0$,

6.8b $\quad B_{sj} = -j_s \eta$, $C_{sj} = 0.$

7. Outline of convergence proof. The proof described in this section applies to any truncated problem satisfying assumptions (i)-(iv) and $(v)_4$ of section 1 and $(i)_1$, $(iii)_1$, $(iv)_1$, $(iv)_y$ of section 6. For such a problem, the solutions of the difference equations, $u_L^{\xi,\eta,\tau}(x,y,t)$, are equibounded and, for any $\varepsilon > 0$, satisfy Lipschitz conditions in $S_T - S^{(\varepsilon)}$ with respect to x_r, y_s, t, $r, s = 1,\ldots,d$, which are uniform in this set for all possible x,y,t,ξ,η,τ,L. This will be proved in sections 8 and 10 below. Because of the uniform bounds and the uniform Lipschitz conditions, from any infinite set of lengths L tending to infinity and of suitable mesh widths ξ, η, τ tending to zero, sequences ξ_k,η_k,τ_k,L_k, $k=1,2,\ldots$,

can be selected such that the solutions

$$u^{(k)}(x,y,t) = u_{L_k}^{\xi_k,\eta_k,\tau_k}(x,y,t)$$

converge continuously [1] to a function $u(x,y,t)$ that is Lipschitz continuous, uniformly so in $S_T - S^{(\varepsilon)}$ for any $\varepsilon > 0$. This limit function will be seen to be a strong solution of the problem.

By continuity, u plainly satisfies the requisite boundary and initial conditions, and we shall now see that equations 1.1 hold almost everywhere, as well. Let $v(x,t)$ be a function of class C' in the region

$$x \; \varepsilon \; \mathcal{L} \; , \qquad 0 \leq t \leq T \qquad (T > 0)$$

that vanishes identically in a neighborhood of the boundary of the region, and let $v_i^n = v(\xi\, i, n\tau)$ denote its restriction to the lattice $S^{(k)} = S_{L_k}^{\xi_k,\eta_k,\tau_k}$. On this lattice, consider the difference equations 6.3 determining $u^{(k)}$. We multiply the members of equation 6.3 by v_i^{n+1}, sum over n and i, and eliminate u-differences by Abel's method to obtain

$$- \sum_{n,i} u_{ij}^n \frac{1}{\tau_k}(v_i^{n+1} - v_i^n) - \sum_{n,i,s} \eta_k j_s u_{ij}^{n+1} \frac{1}{\xi_k} \bar{\delta}_{s,j} v_i^{n+1} + \sum_{n,i} c_{ij}^n v_i^{n+1} u_{ij}^n =$$

$$= \sum_{n,i} v_i^{n+1} g_{ij}^n + \sum_{n,i} v_i^{n+1} \quad \eta_k^d \sum_{j'} K_{ijj'}^n \, u_{ij'}^n \quad .$$

We now multiply this by $\xi_k^d \tau_k \eta_k^d$, for arbitrary a and b sum over j such that $j\eta \, \varepsilon \, Q_{a,b}$, and let $k \to \infty$. Continuous convergence, and the remarks in section 2 concerning the improper integral, show that, as the result,

$$-\int_{Q_{a,b}} dy \iint u(v_t + \sum_s y_s v_{x_s} + cv)dxdt = \int_{Q_{a,b}} dy \iint v(g + \int Kudy')dxdt \ .$$

Thus, u satisfies equation 1.1 in a weak, integral sense. For any $\varepsilon>0$, however, u is Lipschitz continuous in $S - S^{(\varepsilon)}$. If $Q_{a,b}$, therefore, intersects no coordinate plane $y_s = 0$, the partial differentiations in the above formula may be transferred to u by integrating by parts, and we thus obtain

$$\int_{Q_{a,b}} dy \iint v \left\{ u_t + \sum_s y_s u_{x_s} + cu - g - \int Kudy' \right\} dxdt = 0 \ ,$$

a formula valid for every function v, and every a and b, as described. It follows that equation 1.1 holds almost everywhere, and, hence, is a strong solution, as contended.

8. A bound for the solution of the difference equations and an estimate for its t-difference quotients. In this section, under the hypotheses, say, of Section 6, we shall first prove

8.1 $\quad |u_L^{\xi,\eta,\tau}(x,y,t)| < M(t) \ ,$

where $M(t) = \phi_o e^{k_o t} + \frac{b_o}{k_o}(e^{k_o t} - 1)$. Then we shall obtain a Lipschitz condition with respect to t of the form

$$8.2 \quad |u_L^{\xi,\eta,\tau}(x,y,t') - u_L^{\xi,\eta,\tau}(x,y,t'')| \leq A(1 + \sum_s |y_s|)\, |t' - t''| \quad ,$$

where A depends on $T, M(T)$, and the constants in hypotheses $(i)_1$, $(iii)_1$, $(iv)_1$, $(v)_4$.

For $0 \leq n\tau \leq T$, set

$$M_n = \max|u_{ij}^n| \quad ,$$

the maximum being taken for $|i| \leq I$, $|j| \leq L$. From the difference equations 6.5, the estimate 5.7 , and assumptions $(i)_o$ and (ii) , we have

$$(1 + \theta \sum_s |j_s|\eta)|u_{ij}^{n+1}| \leq M_n + \theta \sum_s |j_s|\eta\, M_{n+1} + b_o\tau + \tau\, k_o M_n \quad .$$

For suitable determinations of i and j , however, $|u_{ij}^{n+1}| = M_{n+1}$. Hence, we arrive at the inequality

$$M_{n+1} \leq (1 + k_o\tau)M_n + b_o\tau \quad ,$$

from which, by induction,

$$M_n \leq (1+k_o\tau)^n M_o + b_o\tau\left[(1+k_o\tau)^{n-1} + (1+k_o\tau)^{n-2} + \ldots + 1\right]$$

$$< M_o e^{k_o n\tau} + \frac{b_o}{k_o}(e^{k_o n\tau} - 1) \quad .$$

Since $M_o \leq \phi_o$, inequality 8.1 follows.

To prove 8.2, we shall consider, in an analogous way, difference equations, initial values, and boundary values pertaining to the t-difference quotients

$$v_{ij}^n = \frac{u_{ij}^{n+1} - u_{ij}^n}{\tau} .$$

(Difference quotients with respect to x or y cannot be estimated in quite the same way, the first because of the boundary condition, the second because the pertinent difference equations will contain x-difference quotients not known to be bounded.) Boundary values are zero, as before. Initial values satisfy the inequality

$$8.3 \qquad |v_{ij}^o| \leq C_j ,$$

where $C_j = \phi_1 \sum_s |j_s|\eta + b_o + (c_o + k_o)\,\phi_o$, as we easily see after substituting $u_{ij}^1 = w_{ij} + u_{ij}^o$ in equation 6.7 (n=0). The resulting relations may be written, in fact, as

$$(1+ \Theta \sum_s |j_s|\eta)w_{ij} = -\tau c_{ij}^o u_{ij}^o + \Theta \sum_s B_{sj} X_s w_{ij} + \Theta \sum_s C_{sj} X_s^{-1} w_{ij}$$

$$+ \Theta \sum_s B_{sj}(-u_{ij}^o + X_s u_{ij}^o) + \Theta \sum_s C_{sj}(-u_{ij}^o + X_s^{-1} u_{ij}^o)$$

$$+ \tau g_{ij}^o + \tau\, \eta^d \sum_{j'} K_{ijj'}^o\, u_{ij}^o .$$

Hence, for $W_j = \max_i |w_{ij}|$, we have

$$(1 + \Theta \sum_s |j_s|\eta)|w_{ij}| \leq \Theta \sum_s |j_s|\eta\, W_j + C_j\tau ,$$

and therefore

$$W_j \leq C_j \tau ;$$

this was the contention in 8.3 .

Having considered the boundary and initial values of v^n_{ij} , from equations 6.7 we must now derive the difference equations the latter quantities satisfy. These difference equations are easily seen to be again of the form 6.7 , their coefficients, and the kernel in the Riemann sum, being in fact unchanged, and the new inhomogeneous term that takes the place of g^n_{ij} consisting of bounded quantities. Dividing by $1 + C_j$, we obtain a system of relations that we will regard as equations for $V^n_{ij} = v^n_{ij}/(1 + C_j)$; by $(v)_4$ (Section 1), the kernel

$$K^n_{ijj'}(1 + C_{j'})/(1 + C_j)$$

in these equations, acted on by $\eta^d \sum_{j'}$, gives a uniformly bounded result. The boundary values of the V^n_{ij} are zero, their initial values, by 8.3 , bounded. Hence, by the argument by which we proved 8.1 , V^n_{ij} is uniformly bounded, i.e., $(1 + C_j)^{-1}|u^{n+1}_{ij} - u^n_{ij}|/\tau$ is uniformly bounded, at least for $0 \leq n\tau \leq T$. This implies statement 8.2 .

9. <u>Boundary behavior.</u> Here we shall consider how the solutions of the difference equations, and associated sums related to the integrals

$$\int_0^{\varepsilon} |u(x,y,t)|\,dy_r \,, \quad \int_{-\varepsilon}^{0} |u(x,y,t)|\,dy_r \qquad (\varepsilon > 0) ,$$

behave near the boundary faces on which u was prescribed.

The results do not depend on any kind of truncation. We also shall estimate the boundary values of x-difference quotients of the solutions, but these only in truncated problems. Three theorems are formulated, based on the hypotheses of Section 6.

THEOREM 9.1. If $y_r \neq 0$, we have

$$9.1 \qquad |u_L^{\xi,\eta,\tau}(x,y,t)| \leq A_{r,y}|x_r - C_{r,y}| \qquad \text{for } 0 \leq t \leq T \ ,$$

where $A_{r,y} = \max(\phi_1 \ , \ (b_o + k_o M(T))/|y_r|)$. ($C_{r,y}$ is defined in Section 1, M(T) in Section 8.)

Proof: Select any fixed r among the indices from 1 to d. With fixed $T > 0$, for $j_r \neq 0$ set

$$P_{jm} \equiv P_{rjm} \equiv \max_{\substack{i \in \mathcal{J}_{r,j} \\ 0 \leq n \leq T/\tau}} |T_{r,j}^{-m} u_{ij}^n| \ .$$

From equations 6.5, for $(n+1)\tau \leq T$ we obviously have

$$9.2 \qquad (1 + \theta \sum_s |j_s|\eta)|T_{r,j}^{-m}u_{ij}^{n+1}| \leq P_{jm} + \theta \sum_{s \neq r} |j_s|\eta P_{jm}$$

$$+ \theta|j_r|\eta \, P_{j,m-1} + b\tau \ ,$$

where $b = b_o + k_o M(T)$.

Either $|T_{r,j}^{-m}u_{ij}^n|$ reaches its maximum P_{jm} initially, in which case

$$9.3 \qquad P_{jm} = \max_{i \in \mathcal{J}_{r,j}} |T_{r,j}^{-m}u_{ij}^o| \leq \phi_1 m\xi \ ,$$

or, for some value of $n, |T^{-m}_{r,j} u^{n+1}_{ij}|$ will assume this maximum. In the latter case, we can replace $|T^{-m}_{r,j} u^{n+1}_{ij}|$ in 9.2 by P_{jm} and obtain after some cancellations

$$9.4 \qquad \theta |j_r| \eta \, P_{jm} \leq \theta |j_r| \eta \, P_{j,m-1} + b \tau .$$

If $j_r \neq 0$, since $\tau = \theta \xi$ we have

$$P_{jm} \leq P_{j,m-1} + \frac{b \xi}{|j_r| \eta} ,$$

and, by induction,

$$P_{jm''} \leq P_{j,m'} + \frac{b(m'' - m')\xi}{|j_r| \eta} ,$$

provided 9.4 holds for $m'' \geq m > m'$. In the last inequality, replace m'' by m, and let m' be the least possible integer. If $m' = 0$, we have

$$P_{jm} \leq bm\xi / |j_r| \eta ,$$

since $P_{j,0} = 0$ from the boundary condition; this implies 9.1. If $m' > 0$, relation 9.3 must hold for $m = m'$: hence,

$$P_{jm} \leq \phi_1 m' \xi + \frac{b}{|j_r| \eta} (m - m')\xi$$

$$\leq m\xi \cdot \max (\phi_1 , b / |j_r| \eta);$$

this again implies 9.1.

THEOREM 9.2. Let $0 < \alpha < 1$, and let ε be a positive number divisible by η such that

9.5 $$\varepsilon \phi_1 \leq b_o + k_o M(T) .$$

Then

9.6$^+$ $$\eta \sum_{j_r=1}^{\varepsilon/\eta} \max_{(x,t)\varepsilon \mathfrak{X}_T} |u_L^{\xi,\eta,\tau}(x;y_1,\ldots,y_{r-1},j_r\eta,y_{r+1},\ldots,y_d;t)| \leq$$

$$\leq C_\alpha (C + x_r)^\alpha$$

and

9.6$^-$ $$\eta \sum_{j_r=1}^{\varepsilon/\eta} \max_{(x,t)\varepsilon \mathfrak{X}_T} |u_L^{\xi,\eta,\tau}(x;y_1,\ldots,y_{r-1},-j_r\eta,y_{r+1},\ldots,y_d;t)| \leq$$

$$\leq C_\alpha (C - x_r)^\alpha ,$$

where

$$C_\alpha = \theta^{-1} \max(M(T), 2\phi_1 C) + (b_o + k_o M(T)) \, [(2C)^{1-\alpha}(1+\log(1+\theta\varepsilon))$$

$$+ \max_{0\leq\zeta\leq 2C} \zeta^{1-\alpha} \log(1 + \zeta^{-\alpha})] .$$

Proof: The following calculations presume $0 < |j_r|\eta \leq \varepsilon$. Let

$$Q_{jm} = \max\, (P_{jm}, \phi_1 m\, \xi) .$$

We note that

$$\theta |j_r| \eta (\phi_1 m\xi) \leq \theta |j_r| \eta (\phi_1 (m-1)\xi) + b\,\tau ,$$

since $-\theta |j_r| \eta\, \phi_1\, \xi + b\tau = (-|j_r|\eta\, \phi_1 + b)\tau \geq (b - \varepsilon\, \phi_1)\tau \geq 0$

by assumption 9.5; hence,

9.7' $$\theta |j_r| \eta (\phi_1 m\xi) \leq \theta |j_r| \eta\, Q_{j,m-1} + b\,\tau .$$

In the case in which 9.3 holds, therefore, we obviously have

$$9.7'' \qquad \theta|j_r|\eta\, P_{jm} \leq \theta|j_r|\eta\, Q_{j,m-1} + b\,\tau \; .$$

Even when 9.3 fails, since 9.4 then is valid, inequality 9.7" again can be deduced. Thus, 9.7" holds without exception. With 9.7', it proves

$$9.7 \qquad \theta|j_r|\eta\, Q_{jm} \leq \theta|j_r|\eta\, Q_{j,m-1} + b\,\tau \; .$$

We shall rewrite this result as

$$(1 + \theta|j_r|\eta)Q_{jm} \leq Q_{jm} + \theta|j_r|\eta\, Q_{j,m-1} + b\,\tau \; .$$

Setting $\alpha_j \equiv \alpha_{r,j} \equiv (1 + \theta|j_r|\eta)^{-1}$, we thus have

$$9.8 \qquad Q_{jm} \leq \alpha_j Q_{jm} + (1 - \alpha_j)Q_{j,m-1} + \alpha_j b\,\tau \; .$$

Now define

$$A_{pm} = \eta \sum_{j_r=1}^{\varepsilon/\eta} \alpha_j^p Q_{jm} \, .$$

To prove 9.6 , it suffices to estimate A_{om} appropriately. Multiplying 9.8 by $\eta\,\alpha_j^p$, $p = 0,1,\ldots,$ and summing over j_r gives us

$$A_{om} \leq A_{1m} + A_{o,m-1} - A_{1,m-1} + b\xi \log(1 + \theta\,\varepsilon)$$

$$A_{pm} < A_{p+1,m} + A_{p,m-1} - A_{p+1,m-1} + b\xi/p \, , \quad p = 1,2,\ldots,$$

since

$$\eta \sum_{j_r=1}^{\varepsilon/\eta} a_j^{p+1} < \int_0^{\varepsilon} \frac{dy}{(1+\theta y)^{p+1}} = \frac{1}{\theta} \log (1 + \theta\varepsilon) \quad \text{for} \quad p = 0$$

$$< \frac{1}{p\theta} \quad \text{for} \quad p = 1,2, \ldots .$$

We rewrite the last results as

$$9.9_o \qquad A_{om} - A_{o,m-1} \leq A_{1m} - A_{1,m-1} + b\xi \log (1 + \theta\varepsilon)$$

and

$$9.9_p \qquad A_{pm} - A_{p,m-1} < A_{p+1,m} - A_{p+1,m-1} + b\xi/p , \quad p = 1,2, \ldots .$$

Noting that $A_{oo} = A_{po} = 0$, $p = 1,2,\ldots,$ by summing with respect to m we have

$$9.10_o \qquad A_{om} \leq A_{1m} + bm\xi \log (1+\theta\varepsilon)$$

and

$$9.10_p \qquad A_{pm} < A_{p+1,m} + bm\xi/p , \quad p = 1,2,\ldots .$$

Hence,

$$9.11 \qquad A_{om} < A_{p+1,m} + bm\xi \left(\log (1+\theta\varepsilon) + 1 + \frac{1}{2} + \ldots + \frac{1}{p}\right) .$$

We now note the estimate

$$A_{p+1,m} \leq Q \int_0^{\varepsilon} \frac{dy}{(1+\theta y)^{p+1}} < Q/\theta p$$

(this assumes $p \geq 1$), where $Q = \max (M(T), 2\phi_1 C)$. Also, we recall the fact, known from Euler, that the expressions

$$1 + \frac{1}{2} + \dots + \frac{1}{p} - \log p$$

are positive and less than 1. From 9.11 we thus have

9.12 $$A_{om} < Q/\Theta p + bm\xi \,(\log p + \log (1 + \Theta\varepsilon) + 1) \quad .$$

Now choose p such that

$$\frac{1}{p} \leq (m\xi)^{\alpha} < \frac{1}{p-1} \quad \text{if} \quad 0 < m\xi < 1 \quad ,$$

$$p = 1 \quad \text{if} \quad m\xi \geq 1 \quad .$$

We thus also have that

$$\log p < \log (1 + (m\xi)^{-\alpha}) \quad ;$$

hence, inequality 9.12 leads to

$$A_{om} < \frac{Q}{\Theta}(m\xi)^{\alpha} + b_{\alpha}(m\xi)^{\alpha} \quad ,$$

where $b_{\alpha} = b(2C)^{1-\alpha}(1+\log (1+\Theta\varepsilon)) + b \max\limits_{0\leq\zeta\leq 2C} \zeta^{1-\alpha} \log (1 - \zeta^{-\alpha})$.

This result implies 9.6.

THEOREM 9.3. For a particular index r, suppose $K(x,y,t,y')$ and $g(x,y,t)$ to be zero when $|y_r| \leq \omega$, ω being, say, an exact positive multiple of η. Then for $j_r \neq 0$, $i \in \vartheta_{r,j}$, $0 \leq \eta\tau \leq T$, 9.13 $$\left| \frac{(T_{r,j}^{-1} - 1)u_{ij}^{n}}{\xi} \right| \leq A_{\omega} ,$$

where $A_{\omega} = \max\left(\phi_1 , \; (b_o + k_o M(T))/\omega\right)$.

Proof: Take $0 < |j_r|\eta \leq \omega$ and $m = 1$ in 9.2. Since $P_{j,m-1} = P_{j,0} = 0$, and, under present hypotheses, b can be replaced by zero, we obtain

9.14 $$(1 + \theta \sum_{s} |j_s|\eta)|T_{r,j}^{-1}u_{ij}^{n+1}| \leq (1 + \theta \sum_{s \neq r} |j_s|\eta)P_{j1} .$$

Either $|T_{r,j}^{-1}u_{ij}^{n}|$ reaches its maximum P_{j1} initially, in which case

$$P_{j1} = \max_{i \in \mathcal{J}_{r,j}} |T_{r,j}^{-1}u_{ij}^{o}| \leq \phi_1 \xi ,$$

or, for some value of n , $|T_{r,j}^{-1}u_{ij}^{n+1}|$ will assume this maximum . In the latter case, we can replace $|T_{r,j}^{-1}u_{ij}^{n+1}|$ in 9.14 by P_{j1} , and the result will prove $P_{j1} = 0$. Therefore, $P_{j1} \leq \phi_1\xi$ without qualification when $0 < |j_r|\eta \leq \omega$, while Theorem 9.1 applies in the contrary case $|j_r|\eta > \omega$. Thus, 9.13 is completely proved.

10. Lipschitz conditions in truncated problems. Our proof of convergence for truncated problems (section 7), under assumptions (i-iv), $(v)_4$, $(i)_1$, $(iii)_1$, $(iv)_1$, $(iv)_y$ (sections 1 and 6), presumed knowledge of Lipschitz conditions with respect to x_r, y_s, t, $r,s = 1, \ldots, d$, satisfied for

$0 \leq t \leq T$ by $u_L^{\xi,\eta,\tau}(x,y,t)$ uniformly in x,y,t,ξ,η,τ,L. A suitable Lipschitz condition with respect to t already is to be had from section 8. Here, we shall deduce the other Lipschitz conditions needed, proving, namely, that, for $0 \leq n\tau \leq T$, the difference quotients

$$\xi^{-1}|\delta_r u_{ij}^n| \;, \quad \eta^{-1}|\varepsilon_r u_{ij}^n|$$

are bounded uniformly with respect to ξ, η, τ, L and all indices.

Let ω describe the truncation: i.e., suppose $K(x,y,t,y')$ and $g(x,y,t)$ to be zero if, for any s, $|y_s| \leq \omega$; $\omega > 0$. First considering differences with respect to x , i.e., i-differences, we shall prove

$$10.1 \quad \xi^{-1}|\delta_r u_{ij}^n| \leq e^{k_o n\tau} \max(\phi_1, A_\omega) + \frac{b_1 + M(n\tau)(c_1+k_1)}{k_o}(e^{k_o n\tau} - 1) ,$$

A_ω being the constant defined in Theorem 9.3.

Estimate 10.1 will be obtained from difference equations for the i-differences considered, which we may obtain by applying δ_r, defined in 5.1, to both members of equations 6.7 . We use the identity

$$\delta_r(a_i v_i) = a_i \, \delta_r v_i + \mathfrak{X}_r v_i \cdot \delta_r a_i ,$$

valid for any quantities a_i and v_i indexed with i, to obtain

$$(1 + \Theta \sum_s |j_s|\eta)\xi^{-1}\,\delta_r u_{ij}^{n+1} = (1 - \tau c_{ij}^n)\xi^{-1}\,\delta_r u_{ij}^n$$

10.2
$$+ \Theta \sum_s B_{sj} X_s\, \xi^{-1}\delta_r u_{ij}^{n+1} + \Theta \sum_s C_{sj} X_s^{-1}\, \xi^{-1}\delta_r u_{ij}^{n+1} + \tau\, \xi^{-1}\delta_r g_{ij}^n$$

$$+ \tau\, \eta^d \sum_{j'} K_{ijj'}^n\, \xi^{-1}\delta_r u_{ij'}^n + \tau\, S_{ij}^n \;,$$

where

$$S_{ij}^n = -\,\xi^{-1}(\delta_r c_{ij}^n) X_r u_{ij}^n + \eta^d \sum_{j'} \xi^{-1}(\delta_r K_{ijj'}^n) X_r u_{ij'}^n \;.$$

By hypothesis $(i)_1$ and inequalities 5.8 and 8.1 ,

10.3 $$|S_{ij}^n| \leq (c_1 + k_1) M(n\tau) \;.$$

Because of the existence of the boundary, there are i-differences $\delta_r u_{ij}^{n+1} = u_{ij}^{n+1} - X_r u_{ij}^{n+1}$ in right members of equations 10.2 that never appear on the left. These are of two types, the first being of those with indices $i \in \mathcal{J}_{s,j}$, $s \neq r$; these vanish. The second type is of differences for which i or $X_r i$ is on $\mathcal{J}_{r,j}$. We shall speak of the differences $\delta_r u_{ij}^{n+1}$ of this type, or of the corresponding difference quotients, as being "situated on", or "on", $\mathcal{J}_{r,j}$. Their behavior is controlled by inequality 9.13, which we may reword as

10.4 $$\xi^{-1}|\delta_r u_{ij}^n| \leq A_\omega \text{ on } \mathcal{J}_{r,j} \;.$$

For $0 \leq n\tau \leq T$, set

$$D'_n = \max_{r,i,j} \xi^{-1}|\delta_r u^n_{ij}| \ , \quad D_n = \max (D'_n, A_\omega) \ .$$

From 10.2 and 10.3 we have

10.5 $$(1 + \Theta \sum_s |j_s|\eta)\xi^{-1}|\delta_r u^{n+1}_{ij}| \leq (1 + k_o\tau)D'_n + \Theta \sum_s |j_s|\eta D'_{n+1} + b'\tau$$

where $b' = b_1 + M(n\tau)(c_1 + k_1)$. Let the quantities

$$\xi^{-1}|\delta_r u^{n+1}_{ij}|$$

take their maximum D'_{n+1} say for $r = r'$, $i = i'$, $j = j'$. If $i' \notin \mathcal{J}_{r',j'}$ the quantity 10.6 in the left member of 10.5 may be replaced by D'_{n+1} ; thus we deduce $D'_{n+1} \leq (1+k_o\tau)D'_n + b'\tau$ and

10.6 $$D'_{n+1} \leq (1+k_o\tau)D_n + b'\tau \quad .$$

If, to the contrary, $i' \in \mathcal{J}_{r',j'}$, then by 10.4 $D'_{n+1} \leq A_\omega \leq D_n$: hence, relation 10.6 holds in any event. This relation also being true when the left side is replaced by A_ω, we conclude that

10.7 $$D_{n+1} \leq (1 + k_o\tau)D_n + b'\tau \quad .$$

Reasoning as at the end of section 8 proves from this that

$$D_n \leq e^{k_o n\tau} D_o + \frac{b'}{k_o}(e^{k_o n\tau} - 1) \ ,$$

while $D_o \leq \max (\phi_1, A_\omega)$. Thus, estimate 10.1 is completely justified.

A bound for $\eta^{-1}|\varepsilon_r u^n_{ij}|$ is to be had as follows. First, apply $\eta^{-1}\varepsilon_r$

to equation 6.3 deriving difference equations that the difference quotients $\eta^{-1}\varepsilon_r u_{ij}^n$ satisfy. The coefficients in the difference equations are bounded because of 10.1 . The initial values of the difference quotients are known, and their boundary values are zero. Then arguing as in section 8, from the difference equations deduce recursive inequalities for

$$E_n = \max_{r,i,j} \eta^{-1}|\varepsilon_r u_{ij}^n| \ ,$$

and, from the recursive inequalities, relate E_n to E_o . The result is the desired bound.

11. Hölder conditions in untruncated problems. Proof of Theorem 3. Under hypotheses (i-iv) of section 1, any problem has a weak solution, obtainable (Section 4) as the limit of solutions of truncated problems; if absolutely continuous with respect to t, x_s, $s = 1,\ldots,d$, in the region $S_T - S^{(o)}$, this solution is strong (Section 1). Here, we shall prove absolute continuity when hypotheses $(i)_\alpha$, $(i)_t$, $(iv)_1$, and $(v)_1$ to $(v)_4$ of Section 1 are added to the others. Under these added assumptions we shall show, more specifically, that the (weak) solution of the problem satisfies a Lipschitz condition with respect to t and Hölder conditions with respect to x_s, $s = 1,\ldots,d$, that are uniform in the complement

$$H_\varepsilon : \ |y_s| \geq \varepsilon \ , \quad s = 1,\ldots,d,$$

of an arbitrary ε-neighborhood of the planes $y_s = 0$, $s = 1,\ldots,d$. This solution accordingly being a strong solution of the problem, Theorem 3 will be proved.

As before, we normalize c by the condition $c \geq 0$.

The Lipschitz condition in t is immediate: by Section 8, such a condition holds uniformly in the approximating truncated problems, and the condition of course is preserved in the limit. The Hölder conditions with respect to x remain to be considered. Since such conditions also are preserved under the operation of taking a limit, it will suffice to prove, under the stipulated assumptions, that the solutions $u_L^{\xi,\eta,\tau}(x,y,t)$ of difference equations for <u>truncated</u> problems are subject to Hölder conditions with respect to x_s, $s = 1,\ldots,d$, that, for $y \in H_\varepsilon$ and $0 \leq t \leq T$, are uniform with respect to x,y,t,ξ,η,τ,L. It will be enough, in other words, to find bounds for $|(X_r^m - 1)u_{ij}^n|/(m\xi)^\alpha$ that hold for $n\tau \leq T$, $j\eta\varepsilon H_\varepsilon$, and all ξ,η,τ,L. We shall do so under the restriction, for instance, that $\eta = O(\xi^\alpha)$.

From equations 6.7 we obtain

11.1 $$(1+\theta \sum_s |j_s|\eta)(X_r^m - 1)u_{ij}^{n+1} = (1 - \tau X_r^m c_{ij}^n)(X_r^m - 1)u_{ij}^n$$

$$+ \theta \sum_s B_{sj} X_s (X_r^m - 1)u_{ij}^{n+1} + \theta \sum_s C_{sj} X_s^{-1}(X_r^m - 1)u_{ij}^{n+1}$$

$$+ \tau \eta^d \sum_{|j'|\leq L} X_r^m K_{ijj'}^n \cdot (X_r^m - 1)u_{ij'}^n + \tau R_{ij,rm}^n ,$$

where

$$R_{ij,rm}^n = -(X_r^m - 1)c_{ij}^n \cdot u_{ij}^n + (X_r^m - 1)g_{ij}^n + \eta^d \sum_{|j'|\leq L} (X_r^m - 1)K_{ijj'}^n \cdot u_{ij'}^n .$$

Because of the existence of the boundary, there are differences $(X_r^m - 1)u_{ij}^{n+1}$ in right members of the above equations that never appear on the left. These are of two types, the first consisting of those for which

$i \in \mathcal{J}_{s,j}$, $s \neq r$; by the boundary condition, these vanish. The second type of difference is that for which i or $X_r i$ is on $\mathcal{J}_{r,j}$; Theorem 9.1 estimates this second kind of difference, but not uniformly: hence, reasoning of the type of section 10 cannot be used unmodified.

For $0 \leq n\tau \leq T$, let

$$\bar{\Delta}^n_{rjm} = \max\nolimits_i' \left|(X^m_r - 1)u^n_{ij}\right| ,$$

where $\max_i'$ indicates a maximum taken for such indices i as do not correspond to a boundary value (a value of one of the two kinds discussed above) of the difference concerned. The boundary values are estimated by P_{rjm} , defined in section 9. Hence, for the quantity

$$\Delta^n_{rjm} = \max\left(\bar{\Delta}^n_{rjm} , P_{rjm}\right) \quad \text{when} \quad j_r \neq 0$$

$$= \bar{\Delta}^n_{rjm} \quad \text{when} \quad j_r = 0$$

we have

11.2 $$\left|(X^m_r - 1)u^n_{ij}\right| \leq \Delta^n_{rjm}$$

for all values of the various indices.

A uniform estimate for Δ^n_{rjm} (of the form $\Delta^n_{rjm} \leq \text{constant } (m\xi)^{\alpha}$) , according to a previous remark, is out of the question. For any $\varepsilon > 0$ satisfying condition 9.5 , such an estimate will be possible, however, for

$$E^n_{\varepsilon rm} = \max_{\substack{\varepsilon \leq |j_r| \eta \leq L\eta \\ 0 \leq |j_s| \leq L \\ (s \neq r)}} \Delta^n_{rjm} .$$

The process of estimation will involve, as auxiliary quantities besides the latter,

$$I^n_{rm+} = \max_{\substack{|j_s| \leq L \\ (s \neq r)}} \eta \sum_{j_r=1}^{\varepsilon/\eta} \Delta^n_{rjm}$$

$$I^n_{rm-} = \max_{\substack{|j_s| \leq L \\ (s \neq r)}} \eta \sum_{-j_r=1}^{\varepsilon/\eta} \Delta^n_{rjm}$$

$$I^n_{rm} = \max (I^n_{rm+} , I^n_{rm-}) \quad .$$

Concerning the initial values of the new quantities, it is pertinent to state

11.3a $\quad E^o_{\varepsilon rm} \leq \max (\phi_1, (b_o + k_o M(T))/\varepsilon) \quad m\xi \quad ,$

as follows from Theorem 9.1 and hypothesis $(iv)_1$, and

11.3b $\quad I^o_{rm} \leq \varepsilon\phi_1 \, m\xi \; + \; C_\alpha (m\xi)^\alpha \quad ,$

a consequence of Theorem 9.2.

The estimation process begins with the observations that

$$|R^n_{ij,rm}| \leq c'(m\xi)^\alpha \quad ,$$

where. $c' = M(T)(c_\alpha + k_\alpha) + b_\alpha$, and that

$$|\eta^d \sum_{|j'| \leq L} X^m_r K^n_{ijj'} \cdot (X^m_r - 1) u^n_{ij'}| = |\eta^d \sum_{\substack{j'_s \\ (s \neq r)}} (\sum_{0 < |j'_r| \eta \leq \varepsilon} + \sum_{|j'_r| \eta > \varepsilon}$$

$$+ \text{ term with } j'_r = 0)| \leq 2k \, I^n_{rm} + k_o \, E^n_{\varepsilon rm} + 2k\eta \, M(T) \; .$$

From these and 11.2 , equation 11.1 easily gives us the preliminary inequality

$$(1+\theta \sum_s |j_s| \eta)\ \bar{\Delta}^{n+1}_{rjm} \leq \Delta^n_{rjm} + \theta \sum_s |j_s| \eta\ \Delta^{n+1}_{rjm} + 2k\tau I^n_{rm} + 2\tau\eta k\ M(T)$$

$$+ k_o\ \tau E^n_{\varepsilon rm} + c'\tau(m\xi)^\alpha .$$

If, when $j_r \neq 0$. $\bar{\Delta}^{n+1}_{rjm}$ be replaced by P_{rjm} , the inequality remains good. Hence, $\bar{\Delta}^{n+1}_{rjm}$ can be replaced by Δ^{n+1}_{rjm} . Making the replacement and a consequent cancellation, we arrive at

$$11.4 \quad \Delta^{n+1}_{rjm} \leq \Delta^n_{rjm} + \tau(2k\ I^n_{rm} + k_o\ E^n_{\varepsilon rm} + c'(m\xi)^\alpha + 2\eta\ k\ M(T)) \ .$$

Summing appropriately over j_r , we now easily deduce

$$I^{n+1}_{rm+} \leq I^n_{rm+} + \tau(2k\ \varepsilon\ I^n_{rm} + \varepsilon k_o\ E^n_{\varepsilon rm} + \varepsilon c'(m\xi)^\alpha + 2\eta\varepsilon\ k\ M(T))$$

and a similar inequality concerning I^n_{rm-} ; from both, we have

$$11.5 \quad I^{n+1}_{rm} \leq (1+2k\tau\varepsilon)I^n_{rm} + \tau\varepsilon(k_o\ E^n_{\varepsilon rm} + c'(m\xi)^\alpha + 2\eta\varepsilon\ k\ M(T)) \ .$$

From 11.4 we also have

$$11.6 \quad E^{n+1}_{\varepsilon rm} \leq (1+k_o\tau)E^n_{\varepsilon rm} + \tau(2k\ I^n_{rm} + c'(m\xi)^\alpha + 2\eta\varepsilon\ k\ M(T)) \ .$$

Hence, for

$$F_n = F^n_{rm} = \max\ (I^n_{rm}, E^n_{\varepsilon rm})$$

we can deduce the recursive inequalities

$$F_{n+1} \leq (1 + a\tau)F_n + \tau[c'(m\xi)^{\alpha} + 2\eta\varepsilon k \quad M(T)] \quad ,$$

where $a = (k_o + 2k)\max(1,\varepsilon)$. By an inductive process like that of section 8,

$$F_n \leq e^{an\tau}F_o + \left(\frac{e^{an\tau} - 1}{a}\right) [c'(m\xi)^{\alpha} + 2\eta\varepsilon k \quad M(T)] \quad ,$$

while F_o can be estimated from 11.3 . The result is a bound for $F_n/(m\xi)^{\alpha}$, and thus for $\Delta^n_{\varepsilon rm}/(m\xi)^{\alpha}$, depending only on the constants of the problem.

NOTES

[p.37] 1/ See the discussion and references in Section 6, Douglis, On discontinuous solution of quasi-linear, first order partial differential equations, NAVORD Report 6775, U. S. Naval Ordnance Lab., White Oak, Md., 22 Jan., 1960.

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